Chapter 12

Vector valued functions and motions in space

12.1 Curves and Tangents

Definition 12.1.1. A **curve**(or **path**) can be represented as a function $\mathbf{r} : I = [a, b] \to \mathbb{R}^n, n = 2, 3$, called a **parameterized curve**.

A parameterized curve $\mathbf{r}$ in $\mathbb{R}^n$ can be also written as

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}. \tag{12.1}$$

$f(t), g(t), h(t)$ are called **component functions**.

We define the limit of a vector function as

$$\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{L} = (\lim_{t\to t_0} f(t), \lim_{t\to t_0} g(t), \lim_{t\to t_0} h(t)).$$

Definition 12.1.2. It is called differentiable at t, if the limit

$$\mathbf{r}'(t) = \lim_{\Delta t\to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = (f'(t), g'(t), h'(t)) \tag{12.2}$$

exists at t.

The geometric meaning of derivative of $\mathbf{r}(t)$

When $\mathbf{r}'(t) \neq 0$, it represents a **tangent vector** at t.

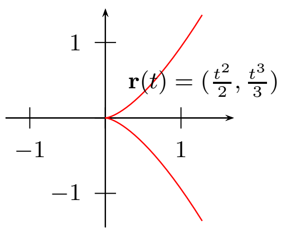


Figure 12.1: At a cusp, $\frac{d\mathbf{r}(t)}{dt}|_{t=0} = 0$

Definition 12.1.3. A curve $\mathbf{r}(t)$ is said to be **smooth** if $d\mathbf{r}/dt$ is continuous and never zero. On a smooth curve, there is no sharp corner or cusps.

Derivatives and Motion

Example 12.1.4. The image of C^1-curve is not necessarily "smooth". It may have sharp edges; (Fig 12.1).

(1) Cycloid: $\mathbf{c}(t) = (t - \sin t, 1 - \cos t)$ has cusps when it touches x-axis. That is, when $\cos t = 1$ or when $t = 2\pi n, n = 1, 2, 3, \cdots$.

(2) Consider $\mathbf{r}(t) = (\frac{t^2}{2}, \frac{t^3}{3})$. Eliminating t, we get

$$(2x)^3 = (3y)^2.$$

We see $\frac{d\mathbf{r}(t)}{dt} = (t^3, t^2)|_{t=0} = 0$ and from Figure 12.1 we see it has a cusp when $t = 0$.

At all these points, we can check that $\mathbf{c}'(t) = 0$.(Roughly speaking, tangent vector has no direction or does not exist.)

12.2 Arc Length

Definition 12.2.1 (Arc Length)**.** Suppose a curve C has one-to-one differentiable parametrization $\mathbf{r}$. Then the **arc length** is defined by

$$L(\mathbf{r}) = \int_a^b \|\mathbf{v}(t)\|\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$$

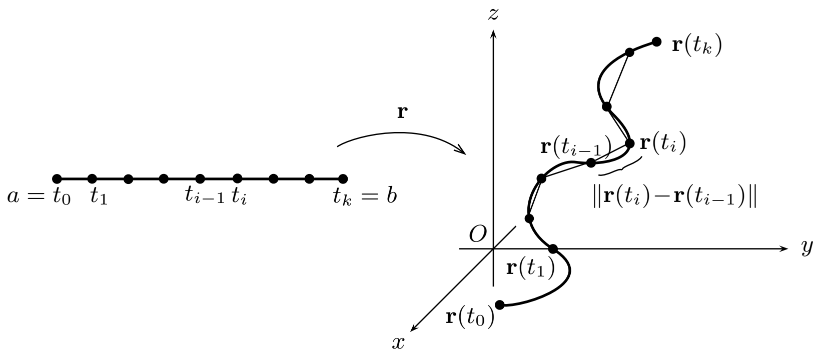


Figure 12.2: Riemann sum of the curve length

The sum of the line segment is

$$\begin{aligned}\sum_{i=1}^{k} \Delta s_i &= \sum_{i=1}^{k} \|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})\| \\ &= \sum_{i=1}^{k} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2} \\ &= \sum_{i=1}^{k} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta z_i}{\Delta t_i}\right)^2} \Delta t_i.\end{aligned}$$

As $k \to \infty$ it converges to

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt. \tag{12.3}$$

Velocity and speed

Assume the path $\mathbf{r}(t) = (x(t), y(t), z(t))$ represents the movement of an object.

Then the **velocity** at $t = t_0$ is given as

$$\mathbf{r}'(t_0) = \lim_{h \to 0} \frac{\mathbf{r}(t_0 + h) - \mathbf{r}(t_0)}{h} = (x'(t_0), y'(t_0), z'(t_0)).$$

Example 12.2.2. If an object follow moving along the curve $\mathbf{c}(t) = t\mathbf{i} + t^2\mathbf{j} + e^t\mathbf{k}$ at time t takes off the curve at $t = 2$ and travels for 5 seconds. Find the location.

sol. We assume the object travels along the tangent line after taking off the curve. The velocity at $t = 2$ is $\mathbf{c}'(2) = \mathbf{i} + 4\mathbf{j} + e^2\mathbf{k}$. Hence the location 5 second after taking off the curve

$$\begin{aligned}\mathbf{c}(2) + 5\mathbf{c}'(2) &= 2\mathbf{i} + 4\mathbf{j} + e^2\mathbf{k} + 5(\mathbf{i} + 4\mathbf{j} + e^2\mathbf{k}) \\ &= 7\mathbf{i} + 24\mathbf{j} + 6e^2\mathbf{k}.\end{aligned}$$

Hence the location is $(7, 24, 6e^2)$.

□

Arc-Length Parameter

Recall : Given a C^1-parametrization of a curve $C : [a, b] \to \mathbb{R}^3$. Then we have seen that the **arc length** of C is given by

$$L(\mathbf{r}) = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$$

Definition 12.2.3. Now we fix a base point $P = P(t_0)$ and let the upper limit be the variable t. Then the arclength becomes a function of t, called the **arc-length function** :

$$s(t) = \int_{t_0}^t \|\mathbf{r}'(\tau)\|\, d\tau.$$

The arc-length (parameter)function satisfies

$$\frac{ds}{dt} = s'(t) = \|\mathbf{r}'(t)\| = speed.$$

We assume $\mathbf{r}'(t) \neq 0$ so that $\frac{ds}{dt}$ is always positive. Then we can solve for s in terms of t. Hence we can use s as a new parameter.

Example 12.2.4. For the helix $\mathbf{r}(t) = (a\cos t, a\sin t, bt)$, we can find a new parametrization by s as follows:

$$s(t) = \int_0^t \|\mathbf{r}'(\tau)\|\, d\tau = \int_0^t \sqrt{a^2 + b^2} d\tau = \sqrt{a^2 + b^2}\ t,$$

so that

$$s = \sqrt{a^2 + b^2}\ t, \text{ or } t = \frac{s}{\sqrt{a^2 + b^2}}.$$

Hence

$$\mathbf{r}(t(s)) = \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}}\right).$$

Definition 12.2.5. The **unit tangent vector T** of the path $\mathbf{r}$ is the normalized velocity vector

$$\mathbf{T} = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

Example 12.2.6. For the helix $\mathbf{r} = (a\cos t, a\sin t, bt)$, we have

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}}{\sqrt{a^2+b^2}}.$$

Example 12.2.7. For the curve $\mathbf{r} = (t, t^2, t^3)$, we have

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}.$$

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}}{\sqrt{1+4t^2+9t^4}}.$$

But arclength is not easy to compute:

$$s(t) = \int_0^t \sqrt{1+4t^2+9t^4}\,dt.$$

Example 12.2.8 (Change of the position $\mathbf{r}$ vector w.r.t arclength)**.** Assume $\mathbf{r}(s)$ be a parametrization by arclength parameter. Then by the chain rule and property of arclength parameter, we have

$$\begin{aligned}\mathbf{r}'(t) &= \mathbf{r}'(s)\frac{ds}{dt}\\ &= \mathbf{r}'(s)\|\mathbf{r}'(t)\|.\end{aligned}$$

Since $\|\mathbf{r}'(t)\| \neq 0$, we have

$$\mathbf{r}'(s) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \left(\text{ i.e., } \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} = \mathbf{T}\right).$$

Thus $\mathbf{r}(s)$ has always unit speed (i.e., $\mathbf{r}'(s)$ always has a unit length). The two parametrization $(a\cos t, a\sin t)$ and $(a\cos 2\pi t, a\sin 2\pi t)$ have different speeds along the same circle. For the first one, $\mathbf{r}'(t) = (-a\sin t, a\cos t)$. So

$$s(t) = \int_0^t \sqrt{a^2} d\tau = at.$$

So

$$(a\cos t, a\sin t) = (a\cos\frac{s}{a}, a\sin\frac{s}{a}).$$

While for the second one, $\mathbf{r}'(t) = (-2a\pi\sin t, 2a\pi\cos t)$. So

$$s(t) = \int_0^t 2a\pi d\tau = 2a\pi t.$$

Solving $t = s/2a\pi$. So

$$(a\cos 2\pi t, a\sin 2\pi t) = (a\cos\frac{s}{a}, a\sin\frac{s}{a}).$$

So the parametrization by the arc length parameter is the same. In fact, it is independent of any parametrization(Why?)

12.3 Curvature and Normal vectors of a Curve

To measure how the curve bends we need to define the following:

Definition 12.3.1. The **curvature** of a path $\mathbf{r}$ is the rate of change of unit tangent vector $\mathbf{T}$ per unit of length along the path. In other words,

$$\kappa(t) = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \frac{1}{\|\mathbf{v}\|}\left\|\frac{d\mathbf{T}}{dt}\right\|.$$

Circular Orbits

Consider a particle moving along a circle of radius r_0. We can represent its motion as

$$\mathbf{r}(t) = (r_0\cos t, r_0\sin t).$$

Since speed is $\|\mathbf{r}'(t)\| = v = r_0$. So the motion is described as

$$\mathbf{v} = \mathbf{r}'(t) = (-r_0\sin t, r_0\cos t), \ \|\mathbf{v}\| = r_0.$$

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = (-\sin t, \cos t)$$
$$\frac{d\mathbf{T}}{dt} = (-\cos t, -\sin t)$$
$$\left\|\frac{d\mathbf{T}}{dt}\right\| = 1.$$

Hence

$$\kappa = \frac{1}{\|\mathbf{v}\|} = \frac{1}{r_0} = \frac{1}{radius}.$$

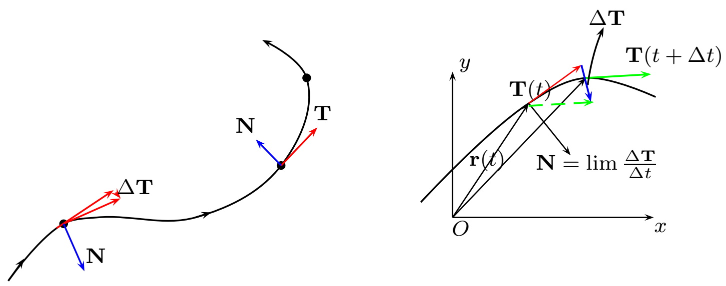


Figure 12.3: **T** turns in the direction of **N**

Since $\mathbf{T}(t)$ is a vector whose length is constant, we have $1 = \|\mathbf{T}(t)\|^2 = \mathbf{T}(t) \cdot \mathbf{T}(t)$. Taking the derivative of constant is zero. Hence

$$0 = \frac{d}{dt}[\mathbf{T}(t) \cdot \mathbf{T}(t)] = \mathbf{T}'(t) \cdot \mathbf{T}(t) + \mathbf{T}(t) \cdot \mathbf{T}'(t) = 2\mathbf{T}(t) \cdot \mathbf{T}'(t).$$

Thus $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$ for all t.

The vector $d\mathbf{T}/ds$ turns in the direction along which the curve turns.

Definition 12.3.2. At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}.$$

The second equality is verified as follows.

$$\begin{aligned}\mathbf{N} &= \frac{d\mathbf{T}/ds}{\|d\mathbf{T}/ds\|} \text{ (use Chain rule)} \\ &= \frac{(d\mathbf{T}/dt)(dt/ds)}{\|d\mathbf{T}/dt\|(dt/ds)} \\ &= \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}.\end{aligned}$$

The vector $\frac{d\mathbf{T}}{ds}$ point in the direction in which $\mathbf{T}$ turns as the curve bends.

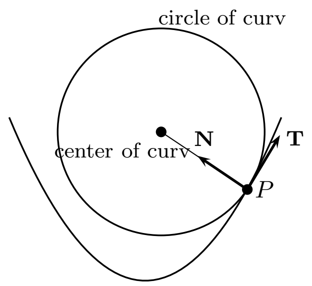


Figure 12.4: Circle of Curvature

Circle of Curvature for Plane curves

The **circle of curvature** or **osculating circle** at a point P is defined when $\kappa \neq 0$. It is a circle that

(1) has the same tangent line as the curve has

(2) has the same curvature as the curve has

(3) has center in the concave side

The **radius of curvature** of the curve at P is the radius of the circle of curvature. (i.e, $1/\kappa$)

Example 12.3.3. Find the osculating circle of parabola $y = x^2$ at the origin.

sol. We parameterize the parabola by

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}.$$

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \\
|\mathbf{v}| &= \sqrt{1+4t^2} \\
\mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} = (1+4t^2)^{-1/2}\mathbf{i} + 2t(1+4t^2)^{-1/2}\mathbf{j}.
\end{aligned}$$

$$\frac{d\mathbf{T}}{dt} = -4t(1+4t^2)^{-3/2}\mathbf{i} + [2(1+4t^2)^{-1/2} - 8t^2(1+4t^2)^{-3/2}]\mathbf{j}.$$

When $t = 0$, $\mathbf{N} = \mathbf{j}$ and

$$\kappa = \frac{1}{|\mathbf{v}(0)|}\left|\frac{d\mathbf{T}}{dt}(0)\right| = \sqrt{0^2+2^2} = 2.$$

the center of the osculating circle is

$$(x-0)^2 + (y-\frac{1}{2})^2 = (\frac{1}{2})^2.$$

□

Curvature and normal vectors for Space curves

Example 12.3.4. For the helix $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}, a, b > 0$.

$$\begin{aligned}
\mathbf{v} &= -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k} \\
|\mathbf{v}| &= \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2} \\
\mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{a^2+b^2}}[-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}].
\end{aligned}$$

Hence

$$\begin{aligned}
\kappa &= \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| \\
&= \frac{1}{\sqrt{a^2+b^2}}\left|\frac{1}{\sqrt{a^2+b^2}}[-(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}]\right| \\
&= \frac{a}{a^2+b^2}\,|[-\cos t\mathbf{i} - \sin t\mathbf{j}]| = \frac{a}{a^2+b^2}.
\end{aligned}$$

Now the normal vector $\mathbf{N} = \mathbf{j}$.

$$\begin{aligned}
\frac{d\mathbf{T}}{dt} &= -\frac{1}{\sqrt{a^2+b^2}}[(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}] \\
|\frac{d\mathbf{T}}{dt}| &= \frac{1}{\sqrt{a^2+b^2}}\sqrt{a^2\cos^2 t + b^2\sin^2 t} = \frac{a}{\sqrt{a^2+b^2}} \\
\mathbf{N} &= \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = -\frac{\sqrt{a^2+b^2}}{a}\frac{1}{\sqrt{a^2+b^2}}[(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}] \\
&= -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}.
\end{aligned}$$

Hence $\mathbf{N}$ is always lying in the xy - plane and pointing toward z axis.

12.4 Tangent and Normal components of a

We define the **binormal** vector $\mathbf{B}$ by

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}.$$

The three vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ form an orthogonal coordinate system (called **TNB frame**.

We see

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt} \\
\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\kappa\mathbf{N}\frac{ds}{dt}\right) \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N} == a_T\mathbf{T} + a_N\mathbf{N}.
\end{aligned}$$

$$\boxed{a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2}} \tag{12.4}$$

Torsion

How does $d\mathbf{B}/ds$ behaves in relation to $\mathbf{T}, \mathbf{N}, \mathbf{B}$?

$$\frac{d\mathbf{B}}{ds} = \frac{d(\mathbf{T}\times\mathbf{N})}{ds} = \frac{d\mathbf{T}}{ds}\times\mathbf{N} + \mathbf{T}\times\frac{d\mathbf{N}}{ds} = 0 + \mathbf{T}\times\frac{d\mathbf{N}}{ds}.$$

Since $d\mathbf{B}/ds$ is orthogonal to $\mathbf{T}$ and $\mathbf{B}$, it is a scalar multiple of $\mathbf{N}$. Hence we have

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

for some scalar τ. This τ is called **torsion** and

$$\boxed{\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}.}$$

(1) $\kappa = |d\mathbf{T}/ds|$ is the rate at which the normal plane turns about the point P as the point moves along the curve.

(2) $\tau = -(d\mathbf{B})/ds)\mathbf{N}$ is the rate at which the osculating plane turns about $\mathbf{T}$ as the point moves along the curve.

Formula for computing the curvature and torsion

$$\begin{aligned}
\mathbf{v}\times\mathbf{a} &= \left(\frac{ds}{dt}\mathbf{T}\right)\times\left[\frac{d^2s}{dt^2}\mathbf{T}+\kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}\right]\\
&= \left(\frac{ds}{dt}\frac{d^2s}{dt^2}\right)(\mathbf{T}\times\mathbf{T})+\kappa\left(\frac{ds}{dt}\right)^3(\mathbf{T}\times\mathbf{N})\\
&= \kappa\left(\frac{ds}{dt}\right)^3\mathbf{B}.
\end{aligned}$$

Hence

$$|\mathbf{v}\times\mathbf{a}| = \kappa\left|\frac{ds}{dt}\right|^3|\mathbf{B}| = \kappa|\mathbf{v}|^3.$$

$$\boxed{\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}} \tag{12.5}$$

Chapter 14

Partial Derivative

14.1 Functions of several variables

Definition 14.1.1. Let $\mathbf{x}_0 \in \mathbb{R}^n$. The **open ball** (or disk) of radius r with center $\mathbf{x}_0$ is the set of all points $\mathbf{x}$ such that $\|\mathbf{x} - \mathbf{x}_0\| < r$. This is denoted by $B_r(\mathbf{x}_0)(D_r(\mathbf{x}_0))$ or $B(\mathbf{x}_0; r)$. A closed ball is a set of the form $\|\mathbf{x} - \mathbf{x}_0\| \le r$.

Definition 14.1.2 (Interior, Open sets)**.** Let $R \subset \mathbb{R}^n$. A point $\mathbf{x} \in \mathbb{R}^n$ is called an **interior point** of R if there is disk about $\mathbf{x}$ completely contained in R. The set of all interior points of R is said to be **interior** of R.

A set $R \subset \mathbb{R}^n$ is said to be **open** if every point $\mathbf{x}_0 \in R$ is an interior point, i.e., there exists some $r > 0$ such that $B_r(\mathbf{x}_0)$ is contained in R(in symbol, $B_r(\mathbf{x}_0) \subset R$).) Finally, a **neighborhood** of a point $x \in R$ is an open set containing x and contained in R.

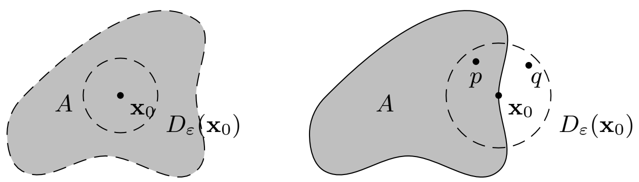


Figure 14.1: Interior point and boundary point: any neighborhood $D_\epsilon(\mathbf{x}_0)$ of a boundary point $\mathbf{x}_0$ contains both points of A and points not in A

Graphs, Level Curves and Contours of functions

Definition 14.1.3. The **graph** of a function is the set

$$graph(f) = \{(\mathbf{x}, f(\mathbf{x})) \in \mathbb{R}^{n+1} \mid \mathbf{x} \in D \subset \mathbb{R}^n\}.$$

Definition 14.1.4. The **level set** of $f\colon \mathbb{R}^n \to \mathbb{R}$ is the set of all $\mathbf{x}$ where the function f has constant value:

$$S_c = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = c, c \in \mathbb{R}\}.$$

Definition 14.1.5. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$. The **section of the graph of f by the plane** $x = c$ is the set of all points (x, y, z), where $z = f(c, y)$, i.e,

$$\{(x, y, z) \in \mathbb{R}^3 \mid z = f(c, y)\}.$$

Limits

Definition 14.1.6 (Limit using ε-δ)**.** Let $\mathbf{f}\colon D \subset \mathbb{R}^n \to \mathbb{R}^m$. We say the **limit** of $\mathbf{f}$ at $\mathbf{x}_0 \in \mathbb{R}^n$ is $\mathbf{L}$, if for any $\varepsilon > 0$ there exists some positive δ such that for all $\mathbf{x} \in D$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$, the inequality $\|\mathbf{f}(\mathbf{x}) - \mathbf{L}\| < \varepsilon$ holds. We write

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{L}.$$

Example 14.1.7 (Two-path test for nonexistence of a limit)**.** Let $f : \mathbb{R}^2 - \mathbf{0} \to \mathbb{R}$ be defined by

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \ or \ (\frac{x^2 - y^4}{x^2 + y^4}).$$

Study the behavior near the origin.

sol. This function is undefined at $\mathbf{0} = (0, 0)$. We observe

$$f(x, 0) = \frac{x^2}{x^2} = 1,\ f(0, y) = \frac{-y^2}{y^2} = -1.$$

Hence limit cannot exists.

□

14.2 Partial Derivatives

Definition 14.2.1. Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ be a real valued function. Then the **partial derivative** with respect to i-th variable x_i is:

$$\lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{e}_i)-f(\mathbf{x})}{h} = \frac{\partial f}{\partial x_i}(\mathbf{x}_0)$$

whenever the limit exists.

Example 14.2.2. Find partial derivatives of $g(x,y) = xy/\sqrt{x^2+y^2}$ at $(1,1)$.

sol. First we compute $\frac{\partial g}{\partial x}(1,1)$:

$$\begin{aligned}\frac{\partial g}{\partial x}(1,1) &= \frac{y\sqrt{x^2+y^2}-xy(x/\sqrt{x^2+y^2})}{x^2+y^2}\\ &= \frac{y(x^2+y^2)-x^2y}{(x^2+y^2)^{-3/2}}\\ &= 2^{3/2}.\end{aligned}$$

□

Example 14.2.3. Find partial derivatives at $(0,0)$ of the function defined by

$$f(x,y) = \begin{cases}\frac{3x^2y-y^2}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0).\end{cases}$$

sol. Use definition:

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{x\to 0}\frac{f(h,0)-f(0,0)}{h} = 0\\ \frac{\partial f}{\partial y}(0,0) &= \lim_{y\to 0}\frac{f(0,h)-f(0,0)}{h} = -1.\end{aligned}$$

□

Existence of partial derivatives does not guarantee the continuity

Example 14.2.4. Given

$$f(x,y) = \begin{cases}0, & xy\neq 0\\ 1, & xy=0.\end{cases}$$

We can show that

(1) Both partial derivatives at $(0,0)$ are zero.

(2) Find the limit of f along the line $y = x$.

(3) f is not continuity at $(0,0)$.

Differentiation of a function of several variable

Review: A one variable function $y = f(x)$ is said to be differentiable at a point a if it satisfies((Figure 14.2))

$$\lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x-a)}{x-a} = 0. \tag{14.1}$$

Alternatively, we have

$$\Delta y = f'(x_0)\Delta x + \epsilon \Delta x, \tag{14.2}$$

where $\epsilon \to 0$ as $\Delta x \to 0$.

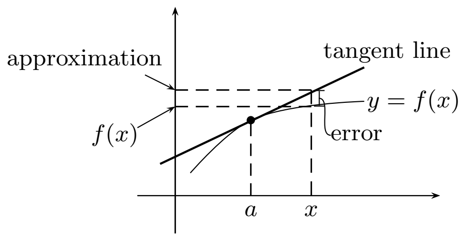


Figure 14.2: tangent approximation of a function of one variable

Definition 14.2.5. We say $f\colon \mathbb{R}^2 \to \mathbb{R}$ is **differentiable** at (a,b) if $\partial f/\partial x$ and $\partial f/\partial y$ exists and for $(x,y) \to (a,b)$, the limit

$$\frac{f(x,y) - f(a,b) - \dfrac{\partial f}{\partial x}(a,b)(x-a) - \dfrac{\partial f}{\partial y}(a,b)(y-b)}{\|(x,y)-(a,b)\|} \to 0.$$

Alternative : $z = f(x,y)$ is **differentiable** at (a,b) if $\partial f/\partial x$ and $\partial f/\partial y$ exists and

$$\Delta z = \frac{\partial f}{\partial x}(a,b)\Delta x + \frac{\partial f}{\partial y}(a,b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \tag{14.3}$$

where ϵ_1 and $\epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. If a function is differentiable at all points of its domain, we say it is **differentiable**.

Definition 14.2.6. In general, Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$. Then we say f differentiable at $\mathbf{a}$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - f(\mathbf{a}) - \left[\frac{\partial f}{\partial x_1}(\mathbf{a}), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{a})\right] \begin{bmatrix} x_1 - a_1 \\ \cdots \\ x_n - a_n \end{bmatrix}}{\|\mathbf{x} - \mathbf{a}\|} = 0.$$

In short,

$$\boxed{\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - f(\mathbf{a}) - \mathbf{D}f(\mathbf{a})(\mathbf{x} - \mathbf{a})}{\|\mathbf{x} - \mathbf{a}\|} = 0.} \tag{14.4}$$

Theorem 14.2.7. *Suppose f_x, f_y are continuous at $\mathbf{x}_0 = (x_0, y_0) \in D$ (an open region). Then (14.3) holds.*

Example 14.2.8. Find the tangent plane of $f(x, y) = x^2 + y^2$ at $(0, 0)$.

sol. We see $(\partial f/\partial x)(0, 0) = (\partial f/\partial y)(0, 0) = 0$ and

$$\begin{aligned} &\lim_{(x,y)\to(0,0)} \frac{f(x, y) - f(0, 0) - \dfrac{\partial f}{\partial x}(0, 0)(x - 0) - \dfrac{\partial f}{\partial y}(0, 0)(y - 0)}{\|(x, y) - (0, 0)\|} \\ &= \lim_{(x,y)\to(0,0)} \frac{f(x, y)}{\|(x, y)\|} = \lim_{(x,y)\to(0,0)} \sqrt{x^2 + y^2} = 0. \end{aligned}$$

Hence it is differentiable at $(0, 0)$. The tangent plane is $z = 0$.

□

Example 14.2.9. Show the function defined by

$$f(x, y) = \begin{cases} \frac{2x^2y^2}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

is differentiable at $(0, 0)$.

sol. It is easy to see that $f_x(0,0) = f_y(0,0) = 0$ by definition. Now

$$\begin{aligned}
&\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - 0 - 0}{\|(x,y) - (0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\|(x,y)\|} \\
&= \lim_{(x,y)\to(0,0)} \frac{2x^2y^2}{(x^2+y^2)^{3/2}} \le \lim_{(x,y)\to(0,0)} \frac{xy(x^2+y^2)}{(x^2+y^2)^{3/2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{xy}{(x^2+y^2)^{1/2}} \\
&\le \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{2(x^2+y^2)^{1/2}} = 0.
\end{aligned}$$

□

Differentiability of vector valued function

Definition 14.2.10. A function $\mathbf{f} = (f_1, \dots, f_m)\colon \mathbb{R}^n \to \mathbb{R}^m$ is said to be **differentiable** at a point $\mathbf{a}$ if all the partial derivatives of $\mathbf{f}$ exists at $\mathbf{a}$,

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - \mathbf{Df}(\mathbf{a})(\mathbf{x}-\mathbf{a})\|}{\|\mathbf{x}-\mathbf{a}\|} = 0.$$

If $m = 1$, then

$$\mathbf{D}f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}.$$

Example 14.2.11. Find the derivative of $\mathbf{Df}(x,y)$.

(1) $\mathbf{f}(x,y) = (xy, x+y)$

(2) $\mathbf{f}(x,y) = (e^{x+y}, x^2+y^2, xe^y)$

sol. (1) $f_1 = xy$, $f_2 = x + y$. Hence

$$\mathbf{Df}(\mathbf{x}) = \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix}.$$

□

Example 14.2.12. Show $f(x,y) = (xy, x+y)$ is differentiable at $(0,0)$.

sol. From example 14.2.11,

$$\mathbf{Df}(0,0) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\lim_{(x,y)\to(0,0)} \frac{\left\| \mathbf{f}(x,y) - \mathbf{f}(0,0) - \mathbf{D}\mathbf{f}(0,0)\begin{bmatrix} x \\ y \end{bmatrix} \right\|}{\|(x,y)-(0,0)\|}$$
$$= \lim_{(x,y)\to(0,0)} \frac{\|(xy, x+y) - (0, x+y)\|}{\|(x,y)\|}$$
$$= \lim_{(x,y)\to(0,0)} \frac{|xy|}{\sqrt{x^2+y^2}} = 0.$$

□

Relation with continuity

Theorem 14.2.13. *If* $\mathbf{f} = (f_1, \ldots, f_n)\colon \mathbb{R}^n \to \mathbb{R}^m$ *has all partial derivatives* $\partial f_i/\partial x_j$ *exist and continuous in a neighborhood of* $\mathbf{x}$*, then* $\mathbf{f}$ *is differentiable at* $\mathbf{x}$*.*

Example 14.2.14. Given

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

Show that

(1) The partial derivatives at $(0,0)$ exist.

(2) f is not differentiable at $(0,0)$.

sol. (1) From definition, we have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{(x,y)\to(0,0)} \frac{f(x,0)-f(0,0)}{x} = 0$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{(x,y)\to(0,0)} \frac{f(0,y)-f(0,0)}{y} = 0.$$

(2) Thus we have $\mathbf{D}f(0,0)$. We consider the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-\mathbf{0}\cdot(x,y)^T}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

Since $\lim_{(x,y)\to(0,0)} xy/(x^2+y^2)$ does not exists, f is not differentiable at $(0,0)$.

□

14.3 Chain rule

Chain rule in several variables

Theorem 14.3.1 (Chain rule-simple). *Suppose* $\mathbf{x}(t) = (x(t), y(t)) \colon \mathbb{R} \to \mathbb{R}^2$ *differentiable at* t_0 *and* $f \colon X \subset \mathbb{R}^2 \to \mathbb{R}$ *differentiable at* $\mathbf{x}_0 = \mathbf{x}(t_0)$ *then the composite function* $h(t) = (f \circ \mathbf{x})(t) \colon \mathbb{R} \to \mathbb{R}$ ($h(t) = f(x(t), y(t))$) *is differentiable at* t_0 *and its derivative* $dh/dt(t_0)$ *is*

$$\frac{dh}{dt}(t_0) = \frac{\partial f}{\partial x}(\mathbf{x}_0)\frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(\mathbf{x}_0)\frac{dy}{dt}(t_0).$$

Proof. We have

$$\begin{aligned}\frac{h(t)-h(t_0)}{t-t_0} &= \frac{f(x(t),y(t)) - f(x(t_0),y(t_0))}{t-t_0} \\ &= \frac{f(x(t),y(t)) - f(x(t_0),y(t)) + f(x(t_0),y(t)) - f(x(t_0),y(t_0))}{t-t_0}.\end{aligned}$$

Let t approach t_0. Then we obtain the result. □

One can use a simpler notation: Let $P_0 = (x_0, y_0) = (x(t_0), y(t_0))$. Then

$$\Delta h = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

As $\Delta x, \Delta y \to 0$, we see

$$\frac{\Delta h}{\Delta t} = f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t} \to f_x \frac{dx}{dt} + f_y \frac{dy}{dt}. \qquad (14.5)$$

Example 14.3.2. Verify the chain rule for $f(x, y) = e^{xy}$ and $\mathbf{x}(t) = (t^2, 2t)$.

Sol. Since $h(t) = f \circ \mathbf{x}(t) = f(x(t), y(t)) = e^{2t^3}$, we have $dh/dt = 6t^2 e^{2t^3}$. On the other hand, by chain rule, we have

$$\frac{dh}{dt} = ye^{xy} \cdot 2t + xe^{xy} \cdot 2 = 6t^2 e^{2t^3}.$$

Example 14.3.3. Let $\mathbf{f} : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be given by $\mathbf{f} = (f_1, \cdots, f_m)$ and $g(\mathbf{x}) = \sin[\mathbf{f}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})]$. Compute $Dg(\mathbf{x})$.

sol. First note that

$$\mathbf{D}g(\mathbf{x}) = \cos[\mathbf{f}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x})]\mathbf{D}[\mathbf{f}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x})].$$

We compute $\mathbf{D}[\mathbf{f}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x})]$ which is

$$\begin{aligned}\mathbf{D}h &= \left[2f_1\tfrac{\partial f_1}{\partial x_1}+\cdots+2f_m\tfrac{\partial f_m}{\partial x_1},\cdots,2f_1\tfrac{\partial f_1}{\partial x_n}+\cdots+2f_m\tfrac{\partial f_m}{\partial x_n}\right]\\ &= 2\mathbf{f}(\mathbf{x})\cdot\mathbf{D}\mathbf{f}(\mathbf{x}),\end{aligned}$$

where $\mathbf{Df}(\mathbf{x})$ is the derivative of $\mathbf{f}$. Finally, we see $\mathbf{D}g(\mathbf{x}) = 2\cos[\mathbf{f}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x})]\mathbf{f}(\mathbf{x})\cdot\mathbf{Df}(\mathbf{x})$.

□

Example 14.3.4 (Polar/Rectangular coordinates conversions)**.** Recall

$$\begin{cases} x = r\cos\theta\\ y = r\sin\theta.\end{cases}$$

Suppose $w = f(x,y)$ is given. We would like view it as a function of (r,θ), i.e,

$$w = g(r,\theta) := f(x(r,\theta), y(r,\theta))$$

and compute $\frac{\partial g}{\partial r}, \frac{\partial g}{\partial \theta}$.

$$\begin{bmatrix}\frac{\partial g}{\partial r} & \frac{\partial g}{\partial \theta}\end{bmatrix} = \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{bmatrix}\begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix} = \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{bmatrix}\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}.$$

Entrywise, we see

$$\begin{cases}\frac{\partial w}{\partial r} &= \cos\theta\frac{\partial w}{\partial x} + \sin\theta\frac{\partial w}{\partial y}\\ \frac{\partial w}{\partial \theta} &= -r\sin\theta\frac{\partial w}{\partial x} + r\cos\theta\frac{\partial w}{\partial y}.\end{cases} \tag{14.6}$$

If we extract the derivative symbol only, we get a differential operator:

$$\begin{cases}\frac{\partial}{\partial r} &= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\\ \frac{\partial}{\partial \theta} &= -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}.\end{cases} \tag{14.7}$$

Similarly, we can show

$$\begin{cases} \frac{\partial}{\partial x} &= \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}. \end{cases} \tag{14.8}$$

Implicit function theorem

Theorem 14.3.5 (Implicit function theorem)**.** *Let $F : X \subset \mathbb{R}^n \to \mathbb{R}$ be class $\mathcal{C}^1$ and let $\mathbf{a}$ be a point of the level set $S = \{\mathbf{x} \in \mathbb{R}^n | F(\mathbf{x}) = c\}$. If $F_{x_n}(\mathbf{a}) \neq 0$, then there is a neighborhood U of $(a_1, a_2, \cdots, a_{n-1})$ in $\mathbb{R}^{n-1}$ and a neighborhood V of a_n in $\mathbb{R}$, and a function $f : U \subset \mathbb{R}^{n-1} \to V$ of class $\mathcal{C}^1$ such that $x_n = f(x_1, x_2, \cdots, x_{n-1})$.*

Example 14.3.6. Consider ellipsoid $x^2/4 + y^2/36 + z^2/9 = 1$. It is the level set of the function

$$F(x, y, z) = \frac{x^2}{4} + \frac{y^2}{36} + \frac{z^2}{9}.$$

At $(\sqrt{2}, \sqrt{6}, \sqrt{3})$, we can check $\frac{\partial F}{\partial z} \neq 0$. Hence z can be solved as function of x and y.

Example 14.3.7. Let $F(x, y, z) = x^2z^2 - y$ and S be the level set of height 0. For points where $F_x = 2z^2 \neq 0$ one can solve for other variables.

14.4 Directional derivatives and Gradient

Definition 14.4.1. Let $\mathbf{u} \in \mathbf{R}^n$ be a unit vector and $\mathbf{a} \in X \subset \mathbf{R}^n$, the **directional derivative** of $f\colon X \to \mathbb{R}$ at $\mathbf{a}$ along $\mathbf{u}$ is $\mathbf{D_u}f(\mathbf{a})$ defined by

$$\frac{d}{dt} f(\mathbf{a} + t\mathbf{u})\Big|_{t=0}.$$

Theorem 14.4.2. *If $f(\mathbf{x})\colon X \subset \mathbb{R}^3 \to \mathbb{R}$ is differentiable and $\mathbf{a} \in X$, then the directional derivative of f at $\mathbf{a}$ along $\mathbf{u}$ exists and is given by*

$$D_{\mathbf{u}} f(\mathbf{a}) = \mathit{grad}\, f(\mathbf{a}) \cdot \mathbf{u} = \nabla f(\mathbf{a}) \cdot \mathbf{u}.$$

Example 14.4.3. Compute the rate of change of $f(x, y, z) = xy - z^2$ at $(1, 0, 1)$ along $(1, 1, 1)$.

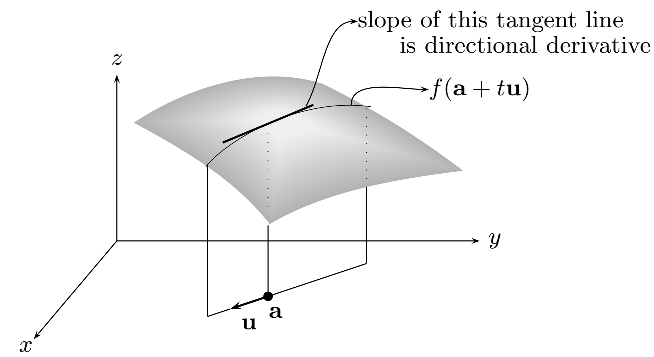


Figure 14.3: Directional Derivative

sol. The unit vector to $(1,1,1)$ is $\mathbf{u} = (1/\sqrt{3})(1,1,1)$. The gradient of f at $(1,0,1)$ is

$$\begin{aligned}\nabla f(1,0,1) &= (f_x, f_y, f_z)|_{(1,0,1)} = (y, x, -2z)|_{(1,0,1)} \\ &= (0,1,-2)\cdot\Big(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\Big) = -\frac{1}{\sqrt{3}}.\end{aligned}$$

□

Direction of steepest ascent(descent)

Gradient is normal to the level set

Consider the level set(surface) of $f(x,y,z)$:

$$S = \{(x,y,z)\in\mathbb{R}^3 \mid f(x,y,z) = k\}.$$

Suppose a curve $\mathbf{c}$ passes the point $\mathbf{x}_0 = (x_0, y_0, z_0)$ lies on the surface S. Then $f(\mathbf{c}(t)) = k$ holds. Then we have by chain rule

$$0 = \frac{d}{dt} f(\mathbf{c}(t)) = \nabla f(\mathbf{c}(t))\cdot\mathbf{c}'(t).$$

Theorem 14.4.4. *Suppose $f(x,y,z)$ is differentiable and $\nabla f(\mathbf{x}_0)\neq 0$. Then $\nabla f(\mathbf{x}_0)$ is normal to the level surface $S = \{(x,y,z)\in\mathbb{R}^3 \mid f(x,y,z) = k\}$.*

14.5 Tangent Plane and differentials

Let us consider the function $z = f(x, y)$.

The direction vector of the tangent line to the y section is $(1, 0, f_x(a, b))$. Similarly, the direction vector of the tangent line to the x- section is $\mathbf{v} := (0, 1, f_y(a, b))$. Hence the tangent plane is determined by he normal vector $\mathbf{N} = \mathbf{u} \times \mathbf{v}$. By computation, we see

$$\mathbf{u} \times \mathbf{v} = -f_x(a, b)\mathbf{i} - f_y(a, b)\mathbf{j} + \mathbf{k}.$$

If (x, y, z) is any point on the plane, then we see $(x - a, y - b, z - f(a, b)) \perp \mathbf{N}$. Hence

$$\boxed{z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).}$$

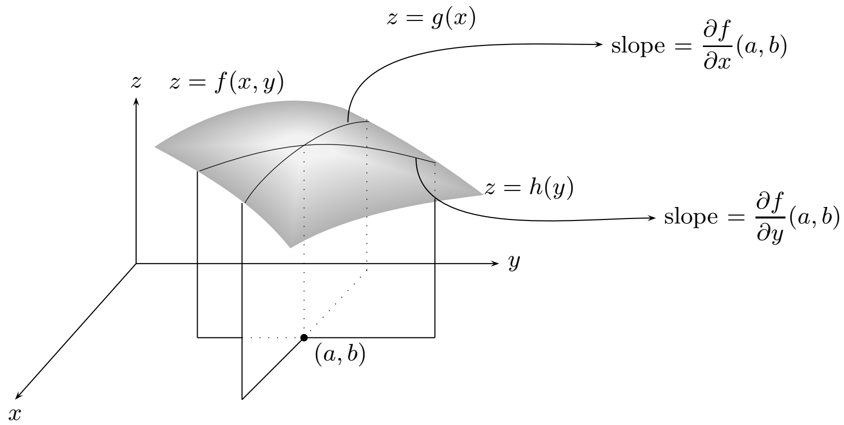


Figure 14.4: Geometric meaning of partial derivative

Definition 14.5.1. The **tangent plane** plane to the surface $S = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = k\}$ at $\mathbf{x}_0 = (x_0, y_0, z_0)$ is given by

$$\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0, \qquad \text{or}$$
$$\frac{\partial f}{\partial x}(\mathbf{x}_0)(x - x_0) + \frac{\partial f}{\partial y}(\mathbf{x}_0)(y - y_0) + \frac{\partial f}{\partial z}(\mathbf{x}_0)(z - z_0) = 0.$$

Compare this with the definition earlier for the graph of $z = f(x, y)$:

$$z - z_0 = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example 14.5.2. Find the equation of the tangent plane to $3xy + z^2 = 4$ at $(1, 1, 1)$.

sol. The gradient $-\nabla f = (3y, 3x, 2z)$ at $(1, 1, 1)$ is $(3, 3, 2)$. Thus the tangent plane is

$$(3, 3, 2) \cdot (x - 1, y - 1, z - 1) = 0.$$

□

Linearization of a function

Suppose f is differentiable near a point (a, b). Then $z = f(x, y)$ satisfies

$$f(x, y) - f(a, b) = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where ϵ_1 and $\epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.

Differential in several variable

Definition 14.5.3. The **total differential of** f, denoted by df is

$$df = \frac{\partial f}{\partial x_1}(\mathbf{x})dx_1 + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{x})dx_n.$$

The significance of differential is that for small $\Delta\mathbf{x}$

$$df \approx \Delta f.$$

Here $\Delta\mathbf{x} = (dx_1, \cdots, dx_n)$ denote small change in the variables and it is also written as $\Delta\mathbf{x} = (\Delta x_1, \cdots, \Delta x_n)$.

Example 14.5.4. Find the differential of $f(x, y, z) = e^{x+y}\sin(yz)$.

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= e^{x+y}\sin(yz)\,dx + e^{x+y}(\sin(yz) + z\cos(yz))\,dy + e^{x+y}y\cos(yz)\,dz. \end{aligned}$$

14.6 Extrema of real valued functions

Local Max, Min

Definition 14.6.1. Let $f(x, y)$ be defined in a region $R \subset \mathbb{R}^n$ containing $\mathbf{a} = (a, b)$. We say

- $f(a,b)$ is a **local maximum** if there is a neighborhood U of $\mathbf{a}$ such that $f(\mathbf{a}) \geq f(\mathbf{x})$ for all $\mathbf{x} \in U$.
- **local minimum** if there is a neighborhood U of $\mathbf{a}$ such that $f(\mathbf{a}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in U$.

Theorem 14.6.2 (First derivative test for local extrema)**.** *If $f\colon R \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$ and assumes an extreme value there, then $\mathbf{D}f(\mathbf{a}) =$ 0.*

Proof. Suppose f has local maximum at $\mathbf{a}$. Then for any $\mathbf{h} \in \mathbb{R}^n$, the function $g(t) = f(\mathbf{a} + t\mathbf{h})$ has a local maximum. Hence

$$g'(0) = \mathbf{D_h}f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{h} = 0$$

Sine this holds for every $\mathbf{h}$, $\nabla f(\mathbf{a}) = \mathbf{0}$. □

Definition 14.6.3. A point $\mathbf{a} \in \mathbb{R}^n$ is called a **critical point** if f is not differentiable or $\nabla f(\mathbf{a}) = \mathbf{0} = (0, \ldots, 0)$.

A critical point $\mathbf{a}$ is called a **saddle point** if for every disk centered at $\mathbf{a}$, there is a point $\mathbf{x}$ where $f(\mathbf{x}) > f(\mathbf{a})$ and a point $\mathbf{x}$ where $f(\mathbf{x}) < f(\mathbf{a})$.

Example 14.6.4. Find critical points of $z = x^2y + y^2x$ and investigate their behavior.

sol. From

$$z_x = 2xy + y^2 = 0, \quad z_y = 2xy + x^2 = 0,$$

we obtain $x^2 = y^2$. For $x = y$, we get $2y^2 + y^2 = 0$ and $(x, y) = (0, 0)$. For $x = -y$, we again get $x = y = 0$. Now for $x = y$, $z = 2x^3$. Not a extreme. So a saddle.

□

Example 14.6.5. Find the extrema of $z = 2(x^2 + y^2)e^{-x^2-y^2}$.

sol.

$$\begin{aligned} z_x &= [4x + 2(-2x)(x^2 + y^2)]e^{-(x^2+y^2)} = 4x(1 - x^2 - y^2)e^{-(x^2+y^2)} \\ z_y &= 4y(1 - x^2 - y^2)e^{-(x^2+y^2)}. \end{aligned}$$

Solving these, we obtain $x = y = 0$ or $x^2 + y^2 = 1$. We can see $(0, 0)$ is a point of local minimum, and the points on the crater's rim are points of local maximum. (Use $t = x^2 + y^2$ so that $z = te^{-t}$ has a local maximum at $t = 1$.)

□

Derivative test for local extreme values

More generally we have second derivative test:

Theorem 14.6.6 (Second derivative test)**.** *Suppose f is $\mathcal{C}^2$ on an open subset U of $\mathbb{R}^2$ and $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ holds, i.e., (x_0, y_0) is a critical point.) Let $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$. Then the following holds:*

(1) f has a local max. if $f_{xx}(x_0, y_0) < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$
(2) f has a local min. if $f_{xx}(x_0, y_0) > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$
(3) f has a saddle point if $f_{xx}f_{yy} - f_{xy}^2 < 0$

Here $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$ is called the **discriminant**.

Example 14.6.7. Find the extrema of $f = x^2 + xy + y^2 + 2x - 2y + 5$.

sol.

First we find the critical point by setting $\mathbf{D}f(x, y) = 0$.

$$\begin{aligned} f_x &= 2x + y + 2 = 0 \\ f_y &= x + 2y - 2 = 0. \end{aligned}$$

Thus $(-2, 2)$ is the only critical point. To determine whether this point is a max or min(or neither), we do as follows:

$$f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = 1,\ D = 2 \cdot 2 - 1^2 > 0.$$

Hence it is a local minimum.

□

Example 14.6.8. Classify the critical points of the following functions.

(1) $g(x, y) = 3x^2 + 6xy + 9y^2$

(2) $g(x, y) = -2x^2 + xy - y^2$

(3) $g(x, y) = x^2 - xy + 2y^2$

sol. All the critical points are $(0, 0)$. For (1), we have $g_x = 6x + 6y$, $g_{xx} = y, g_y = 6x + 18y$, $g_{yy} = 18, g_{xy} = 6$. Hence $D = 6 \cdot 18 - 6^2 = 72 > 0$. Hence $(0, 0)$ is a local min of g.

For (2), we have we have $g_x = -4x + y$, $g_{xx} = -4, g_y = x - 2y$, $g_{yy} = -2, g_{xy} = 1$. Hence $D = (-2)(-4) - 1 = 7 > 0$ and $g_{xx} = -4 < 0$, we see g has local maximum at $(0, 0)$.

For (3), $g_x = 2x - y$, $g_{xx} = 2, g_y = -x + 4y$, $g_{yy} = 4, g_{xy} = -1$. $D = 2 \cdot 4 + 1 = 9 > 0$ and hence g has local minimum at $(0, 0)$.

□

Example 14.6.9. The graph of $g = 1/xy$ is a surface S. Find the point on S closest to $(0, 0)$.

sol. Each point on the surface is $(x, y, 1/xy)$. Hence

$$d^2 = x^2 + y^2 + \frac{1}{x^2y^2}.$$

We find the point which minimize $f(x, y) = d^2(x, y)$ rather than d itself. Solving

$$f_x = 2x - \frac{2}{x^3y^2} = 0, \quad f_y = 2y - \frac{2}{x^2y^3} = 0,$$

we obtain $x^4y^2 = 1$ and $x^2y^4 = 1$. From the first eq. we get $y^2 = 1/x^4$. Substitute into second equation, we get $x^6 = 1$. So $x = \pm 1$ and $y = \pm 1$. Considering the geometry, one can easily see that all these four points give minimum ($d = \sqrt{3}$).(As x or y approaches ∞, $f \to \infty$). So f has no max.

□

Absolute(global) maxima and Minima

Definition 14.6.10. Suppose $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is real valued function. A point $\mathbf{x}_0 \in D$ is a point of **absolute maximum** if $f(\mathbf{x}_0) \geq f(\mathbf{x})$ for all $\mathbf{x} \in D$. Similarly, it is a point of **absolute minimum** if $f(\mathbf{x}_0) \leq f(\mathbf{x})$ for all $\mathbf{x} \in D$.

Strategy of finding absolute(global) maxima and Minima

(1) Find all critical points

(2) Compute values at critical points

(3) Find max or min on the boundary ∂U(by parametrization)

(4) Compare all values obtained in (2) and (3).

Example 14.6.11. Find the maximum and the minimum of $f(x, y) = x^2 + y^2 - x - y + 1$ in $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

sol. First we compute the critical points of f. Since

$$f_x = 2x - 1 = 0, \quad f_y = 2y - 1 = 0,$$

the point $(1/2, 1/2)$ is the only critical point. Since $f_{xx} = 2$, $f_{xy} = 0$, $f_{yy} = 2$, $f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$, $f_{xx} = 2 > 0$, the point $(1/2, 1/2)$ is gives a local minimum by second derivative test. Now check the boundary D: $x^2 + y^2 = 1$. Use parametrization $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$.

$$g(t) = \cos^2 t + \sin^2 t - \cos t - \sin t + 1 = 2 - \cos t - \sin t.$$

Set $g'(t) = \sin t - \cos t = 0$ to get $t = \pi/4, 5\pi/4$ are critical points. We have to check the end points $t = 0, 2\pi$ also. Hence the values are

$$g(0) = 1, \quad g(\pi/4) = 2 - \sqrt{2}.$$

$$g(5\pi/4) = 2 + \sqrt{2}, \quad g(2\pi) = 1.$$

Comparing, we see maximum is at $t = 5\pi/4$, and min at $\pi/4$ on the boundary. Finally compare with $f(1/2, 1/2) = 1/2$. The absolute maximum is $2 + \sqrt{2}$ and absolute minimum is $1/2$.

□

14.7 Constrained Extrema and Lagrange multiplier

Theorem 14.7.1. *Assume $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ are of $\mathcal{C}^1$ class. And the restriction of f to the level set $S = \{\mathbf{x} \in \mathbb{R}^n \mid g(\mathbf{x}) = c\}$ (written as $f|_S$)*

has a (local) maximum or minimum at $\mathbf{x}_0 \in S$ *with* $\nabla g(\mathbf{x}_0) \neq 0$*. Then there is a scalar* λ *(***Lagrange multiplier***) such that*

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0).$$

Example 14.7.2. Let

$$\begin{aligned} A(x,y,z) &= 2(xy+yz)+zx, \\ g(x,y,z) &= xyz-4. \end{aligned}$$

By the Lagrange multiplier method, we have

$$\nabla A = \lambda \nabla g \Rightarrow (2y+z, 2x+2z, 2y+x) = \lambda(yz, zx, xy).$$

This gives three equations in four unknowns, x, y, z and λ. Appending the constraint equation, we have four by four system:

$$\begin{aligned} 2y+z &= \lambda yz \\ 2x+2z &= \lambda zx \\ 2y+x &= \lambda xy \\ xyz &= 4. \end{aligned}$$

Since λ is not essential, we usually eliminate λ using any of the three equations. Thus we get

$$\lambda = \frac{2y+z}{yz} = \frac{2x+2z}{zx} = \frac{2y+x}{xy}.$$

From these we get

$$\frac{2}{z} + \frac{1}{y} = \frac{2}{z} + \frac{2}{x} = \frac{2}{x} + \frac{1}{y}.$$

Hence

$$x = 2y, \quad z = 2y$$

Substituting into last eq. $(2y)y(2y) = 4$. Hence $y = 1, x = z = 2$.

Now a general minimization problem with a constraint is :

Find the minimum of	$f(x,y,z)$
subject to	$g(x,y,z) = c.$

To solve it we solve system of equations with $n+1$ variables

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \tag{14.9}$$

$$g(\mathbf{x}) = c. \tag{14.10}$$

Example 14.7.3. Find max of $f(x,y) = x^2 - y^2$ on $S : x^2 + y^2 = 1$. (See figure) where the two level curves touch.

sol. Since $g(x,y) = x^2 + y^2 = 1$ and $\nabla f = (2x, -2y)$, $\nabla g = (2x, 2y)$, the equation becomes

$$\begin{aligned} f_x(x,y) = \lambda g_x(x,y) &\Longleftrightarrow 2x = \lambda 2x, \\ f_y(x,y) = \lambda g_y(x,y) &\Longleftrightarrow -2y = \lambda 2y, \\ g(x,y) = 1 &\Longleftrightarrow x^2 + y^2 = 1. \end{aligned}$$

From the first equation we get $x = 0$ or $\lambda = 1$. If $x = 0$, we see from third equation $y = \pm 1$. If $\lambda = 1$ then $y = 0$ and $x = \pm 1$. Now

$$\begin{aligned} f(0,1) &= f(0,-1) = -1, \\ f(1,0) &= f(-1,0) = 1. \end{aligned}$$

Hence the max is 1 and the min is -1.

□

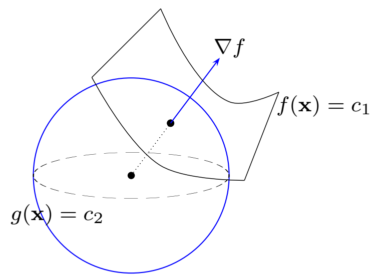


Figure 14.5: Level sets of g meets with the level set of f.

Example 14.7.4. Find max of $f(x,y,z) = x + z$ subject to $x^2 + y^2 + z^2 = 1$.

sol. Let $g(x,y,z) = x^2 + y^2 + z^2$. By the Lagrange multiplier method, we have $\nabla f = \lambda \nabla g$. Thus,

$$
\begin{aligned}
1 &= 2x\lambda, \\
0 &= 2y\lambda, \\
1 &= 2z\lambda, \\
1 &= x^2 + y^2 + z^2.
\end{aligned}
$$

From the first and third equation we see that $\lambda \neq 0$ and $x = z$. Hence from the second equation $y = 0$. Eliminating $\lambda = \frac{1}{2x} = \frac{1}{2z}$, we get $x = z$. From fourth equation we obtain $x = z = \pm 1/\sqrt{2}$. Hence $(1/\sqrt{2}, 0, 1/\sqrt{2})$ and $(-1/\sqrt{2}, 0, -1/\sqrt{2})$.

□

Example 14.7.5. Find extreme points of $f = x + y + z$ subject to $x^2 + y^2 = 2$ and $x + z = 1$.

sol. Constraints are $g_1 = x^2 + y^2 - 2 = 0$ and $g_2 = x + z - 1 = 0$. Thus

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2.$$

Since

$$
\begin{aligned}
g_1 &= x^2 + y^2 - 2 \\
g_2 &= x + z - 1,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
1 &= \lambda_1 \cdot 2x + \lambda_2 \cdot 1 \\
1 &= \lambda_1 \cdot 2y + \lambda_2 \cdot 0 \\
1 &= \lambda_1 \cdot 0 + \lambda_2 \cdot 1 \\
0 &= x^2 + y^2 - 2 \\
0 &= x + z - 1.
\end{aligned}
$$

From third equation we obtain $\lambda_2 = 1$ and so $\lambda_1 \cdot 2x = 0$ and $\lambda_1 \cdot 2y = 1$. From second, we see $\lambda_1 \neq 0$, hence $x = 0$. Thus $y = \pm\sqrt{2}$ and $z = 1$. Hence possible extrema are $(0, \pm\sqrt{2}, 1)$. We can easily check the point $(0, \sqrt{2}, 1)$ gives the maximum value $\sqrt{2} + 1$, and the point $(0, -\sqrt{2}, 1)$ gives minimum $-\sqrt{2} + 1$.

□

14.8 Taylor theorem for two or more variables

14.8.1 Taylor theorem for 2 variables - second order formula

Define a new function $F(t) = f(\mathbf{a}+t\mathbf{h})$. According to Taylor's theorem in one variable, we have for some c,

$$F(t) = F(0) + F'(0)t + \frac{1}{2!}F''(c)t^2.$$

By chain rule we see that

$$F'(t) = f_x\frac{dx}{dt} + f_y\frac{dy}{dt} = hf_x + kf_y = (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})f(\mathbf{a}+t\mathbf{h})$$

Repeating this process, we see

$$F''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(\mathbf{a}+t\mathbf{h}), \cdots, F^{(m)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f(\mathbf{a}+t\mathbf{h}).$$

Expanding in $F(t)$, we see

$$\begin{aligned} F(t) &= F(0) + f_x(\mathbf{a})\frac{dx}{dt} + f_y(\mathbf{a})\frac{dy}{dt} + \frac{1}{2}\left(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right)\big|_{\mathbf{c}} \\ &= f(\mathbf{a}) + hf_x(\mathbf{a}) + kf_y(\mathbf{a}) + \frac{1}{2}\left(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right)\big|_{\mathbf{c}} \end{aligned}$$

Theorem 14.8.1. *(1) The remainder is given by* $R_1(\mathbf{x},\mathbf{a}) = \sum_{i,j=1}^n \frac{1}{2!}f_{x_i,x_j}(\mathbf{c})h_ih_j$ *and* $R_2(\mathbf{x},\mathbf{a}) = \sum_{i,j,k=1}^n \frac{1}{3!}f_{x_i,x_j,x_k}(\mathbf{c})h_ih_jh_k$.

Derivation of second derivative test

(with $h = x-a, k = y-b$) For small value c, it suffices to check(by continuity) the sign of

$$Q(0) = h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b).$$

(1) f has a local max. if $f_{xx}(a,b) < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$
(2) f has a local min. if $f_{xx}(a,b) > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$
(3) f has a saddle point if $f_{xx}f_{yy} - f_{xy}^2 < 0$.

Example 14.8.2. Find 2nd order Taylor approximation of $f(x,y) = e^{x+y}$ near $\mathbf{a} = (0,0)$.

sol.

$$f_x(0,0) = f_y(0,0) = e^0 = 1,$$
$$f_{xx}(0,0) = f_{xy}(0,0) = f_{yy}(0,0) = e^0 = 1.$$

$$f(x,y) = 1 + x + y + \frac{1}{2!}(x^2 + 2xy + y^2) + R_2.$$

As $(x,y) \to (0,0)$, $R_2/\|(x,y)\|^2 \to 0$.

□

Error formula for Taylor expansion

$$f(x,y) = f(a,b) + hf_x(a,b) + kf_y(a,b) + \frac{1}{2}\left(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right)_{\mathbf{c}}.$$

Hence the error of linear approximation can be estimated as

$$|R_1| \le |E_1| = \frac{M}{2}(h+k)^2,\ M = \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\}.$$

$$|R_2| \le |E_2| = \frac{M}{6}(h+k)^3,\ M = \max\{|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\}.$$

14.9 Partial derivatives with constrained variables

Example 14.9.1. Find $\frac{\partial w}{\partial x}$ at $(2,-1,1)$ when $w = x^2 + y^2 + z^2$ and $z^3 - xy + yz + y^3 = 1$ assuming x, y are independent variables.

Chapter 15

Double integral

15.1 Double integral over a rectangle

Assume $R = \{(x, y) \colon a \le x \le b, c \le y \le d\}$ is subdivided into

$$a = x_0 < x_1 < \cdots < x_n = b, \quad c = y_0 < y_1 < \cdots < y_n = d.$$

We call the $\{R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}$ a **partition** of R and let

$$\Delta x_i = x_i - x_{i-1}, \quad \Delta y_j = y_j - y_{j-1}.$$

Definition 15.1.1. Given any function f defined on R, and for any point c_{ij} in R_{ij} consider the **Riemann sum**

$$S = \mathcal{R}(f) = \sum_{i,j=1}^{n} f(c_{ij}) \Delta x_i \Delta y_j, \tag{15.1}$$

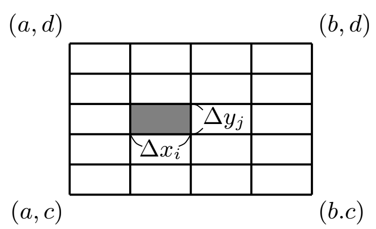


Figure 15.1: A partition of a rectangle

Definition 15.1.2 (Double integral)**.** If the sum S converge to the same limit, then f is called **integrable** over R and we write its limit by

$$\iint_R f(x,y)\,dA = \lim_{\|\mathcal{P}\|\to 0} \sum_{i,j=1}^{n} f(c_{ij})\Delta x_i \Delta y_j.$$

Reduction to iterated integrals

Consider the volume of a solid under f over $R = [a,b] \times [c,d]$. The cross section along $x = x_0$ is the set given by $\{(x_0, y, z) | 0 \le z \le f(x_0, y),\ (c \le y \le d)\}$. The area of cross section is

$$A(x_0) = \int_c^d f(x_0, y)\,dy.$$

Hence by Cavalieri principle, the volume is

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx.$$

The expression on the right hand side is called an **iterated integral**. We change the role of x and y to see

$$\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy.$$

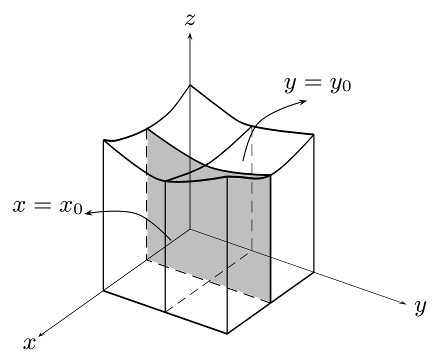


Figure 15.2: Fubini's theorem by Cavalieri Principle

Example 15.1.3. Evaluate

$$\iint_S \cos x \sin y \, dxdy, \quad S = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}].$$

sol.

$$\begin{aligned}\iint_S \cos x \sin y \, dxdy &= \int_0^{\pi/2} \left[\int_0^{\pi/2} \cos x \sin y \, dx \right] dy \\ &= \int_0^{\pi/2} \sin y \left[\int_0^{\pi/2} \cos x \, dx \right] dy = \int_0^{\pi/2} \sin y \, dy = 1.\end{aligned}$$

□

Theorem 15.1.4 (Fubini Theorem 1). *Let f be continuous on $R = [a, b] \times [c, d]$. Then f satisfies*

$$\int_a^b \left[\int_c^d f(x, y) \, dy \right] dx = \int_c^d \left[\int_a^b f(x, y) \, dx \right] dy = \iint_R f(x, y) \, dA. \quad (15.2)$$

Example 15.1.5. Find the volume of the region $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 2 - x - y$.

sol. First fix x. Then the area of cross section with a plane perpendicular to x-axis is

$$A(x) = \int_0^1 (2 - x - y) \, dy.$$

So the volume is

$$\begin{aligned}V &= \int_0^1 A(x) \, dx = \int_{x=0}^{x=1} \int_{y=0}^{y=1} (2 - x - y) \, dydx \\ &= \int_0^1 \left[2y - xy - \frac{y^2}{2} \right]_0^1 dx \\ &= \int_0^1 \left(\frac{3}{2} - x \right) dx = \left[\frac{3x}{2} - \frac{x^2}{2} \right]_0^1 = 1.\end{aligned}$$

□

Example 15.1.6. Compute $\iint_R (x^2 + y) dA$, where $A = [0, 1] \times [0, 1]$.

sol.

$$\iint_R (x^2+y)dA = \int_0^1\int_0^1 (x^2+y)dxdy = \int_0^1 [\int_0^1 (x^2+y)dx]dy = \int_0^1 (\frac{1}{3}+y)dy = \frac{5}{6}.$$

□

15.2 Double integral over general regions

Now we define the integral of more general functions.

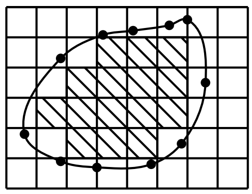


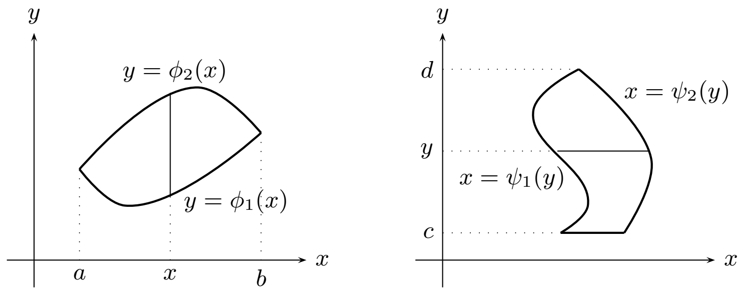
Figure 15.3: Partitioning of nonrectangular region

We only count the sub-rectangles contained in the region:

$$S = \mathcal{R}(f) = \sum f(c_{ij})\Delta A_{ij}. \quad (15.3)$$

If this limit exists as $\|\mathcal{P}\| \to 0$ we define it as a double integral.

Definition 15.2.1. Elementary regions



(a) region of type 1 (b) region of type 2

Figure 15.4: region of type 1, region of type 2

There are three kind of elementary regions: Let $y = \phi_1(x)$, $y = \phi_2(x)$ be two continuous functions satisfying $\phi_1(x) \le \phi_2(x)$ for $x \in [a, b]$. Then the region

$$D = \{(x, y) \mid a \le x \le b,\ \phi_1(x) \le y \le \phi_2(x)\}$$

is called **region of type 1**.

Now change the role of x, y as in figure 15.4 (b). If $x = \psi_1(y)$, $x = \psi_2(y)$, satisfies $\psi_1(y) \le \psi_2(y)$ for $y \in [c, d]$, then the region determined by

$$D = \{(x, y) \mid c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$$

is called **region of type 2**. The region that is both Type 1 and Type 2 is called **region of type 3**. These are called **elementary regions**.

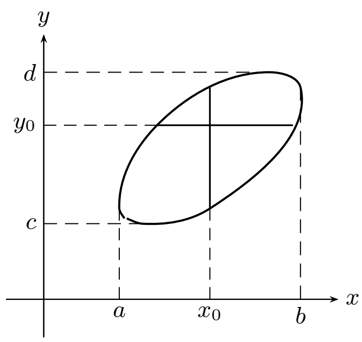


Figure 15.5: Region of type 3

Integrals over elementary regions

Theorem 15.2.2 (Fubini's Theorem (Stronger form)). *Let f be a continuous on an elementary region $D \subset R$.*

(1) If D is a domain of type 1, i.e, $D = \{(x, y) : \phi_1(x) \le y \le \phi_2(x),\ a \le x \le b\}$ for some continuous functions ϕ_1, ϕ_2, then f is integrable on D and

$$\iint_D f(x, y)\, dA = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy \right] dx.$$

(2) Similarly if D is a domain of type 2, i.e, $D = \{(x, y) : \psi_1(y) \le x \le$

$\psi_2(y)$, $c \le y \le d\}$ for some continuous functions ψ_1, ψ_2, then

$$\iint_D f(x,y)\,dA = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\right] dy.$$

Example 15.2.3. Find the following integral when $D: 0 \le x \le 1,\ x \le y \le 1$

$$\iint_D (x+y^2)\,dxdy.$$

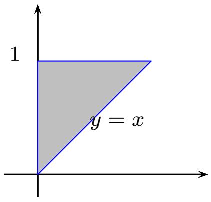


Figure 15.6: Region $0 \le x \le 1,\ x \le y \le 1$

sol. Use Fubini's theorem

$$\begin{aligned}\int_0^1 \int_x^1 (x+y^2)\,dydx &= \int_0^1 \left[xy + \frac{y^3}{3}\right]_x^1 dx \\ &= \int_0^1 \left(x + \frac{1}{3} - x^2 - \frac{x^3}{3}\right) dx \\ &= \left[\frac{x^2}{2} + \frac{x}{3} - \frac{x^3}{3} - \frac{x^4}{12}\right]_0^1 = \frac{5}{12}.\end{aligned}$$

□

Example 15.2.4. Find $\iint_D x^2 y\,dA$ where D is given by $0 \le x,\ 3x^2 \le y \le 4 - x^2$. (Figure 15.7)

sol. Two curves $y = 3x^2$ and $y = 4 - x^2$ meet at the point $(1,3)$. Hence the integral becomes

$$\begin{aligned}\int_0^1 \int_{3x^2}^{4-x^2} x^2 y\,dydx &= \int_0^1 \left(\frac{x^2y^2}{2}\right)\Bigg|_{y=3x^2}^{4-x^2} dx \\ &= \int_0^1 \left(\frac{x^2}{2}((4-x^2)^2 - (3x^2)^2)\right) dx \\ &= \frac{1}{2}\int_0^1 x^2(16 - 8x^2 + x^4 - 9x^4)\,dx = \frac{136}{105}.\end{aligned}$$

□

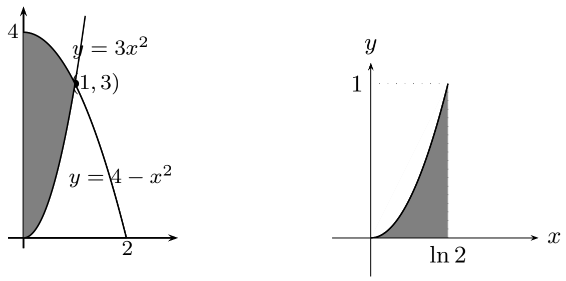


Figure 15.7: Domain of integration of example 15.2.4 and 15.2.6

Example 15.2.5. Find volume of tetrahedron bounded by the planes $y = 0, x = 0, y - x + z = 1$.

sol. We let $z = f(x, y) = 1 - y + x$. Then the volume of tetrahedra is the volume under the graph of f. Hence

$$\begin{aligned} \iint_D (1 - y + x) dA &= \int_{-1}^{0} \int_0^{1+x} (1 - y + x) dy dx \\ &= \int_{-1}^{0} \left[(1 + x)y - \frac{y^2}{2} \right]_{y=0}^{1+x} dx = \frac{1}{6}. \end{aligned}$$

□

Example 15.2.6. Let D be given by $D = \{(x, y) | 0 \le x \le \ln 2, \quad 0 \le y \le e^x - 1\}$. Express the double integral

$$\iint_D f(x, y)\, dA$$

in two iterated integrals.

sol. See figure 15.7. To view it as a region of type 1, the points of intersection is $y = 0$, $y = e^x - 1 (0 \le x \le \ln 2)$. Hence

$$\int_0^{\ln 2} \int_0^{e^x - 1} f(x, y)\, dy dx.$$

As a y-simple region, the points of intersection is $x = \ln(y + 1)$, $x =$

$\ln 2 (0 \le y \le 1)$. So the integral is

$$\int_0^1 \int_{\ln(y+1)}^{\ln 2} f(x,y)\,dxdy.$$

□

15.3 Change order of integration

Suppose D is of type 3. Then it is given by two ways:

$$\phi_1(x) \le y \le \phi_2(x),\ \ a \le x \le b$$

and

$$\psi_1(y) \le x \le \psi_2(y),\ \ c \le y \le d.$$

Thus by Theorem 15.2.2

$$\iint_D f(x,y)dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dydx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dxdy.$$

Example 15.3.1. Compute by change of order of integration

$$\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2-y^2)^{1/2}\,dydx.$$

sol.

$$\begin{aligned}
\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2-y^2)^{1/2}\,dydx &= \int_0^a \int_0^{(a^2-y^2)^{1/2}} (a^2-y^2)^{1/2}\,dxdy \\
&= \int_0^a [x(a^2-y^2)^{1/2}]_0^{(a^2-y^2)^{1/2}} (a^2-y^2)^{1/2}\,dy \\
&= \int_0^a (a^2-y^2)\,dy = \frac{2a^3}{3}.
\end{aligned}$$

□

Example 15.3.2. Find

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y}\,dydx.$$

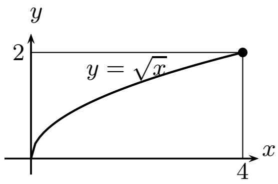


Figure 15.8: Change order of integration

sol. We change the order of integration

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dydx = \int_0^\pi \int_0^y \frac{\sin y}{y}\, dxdy$$
$$= \int_0^\pi \left[\frac{\sin y}{y} x\right]_{x=0}^{x=y} dy$$
$$= \int_0^\pi \sin y\, dy = [-\cos y]_0^\pi = 2.$$

□

Example 15.3.3. Find

$$\int_0^2 \int_{y^2}^4 y\cos(x^2)\, dx\, dy.$$

sol. It is very difficult to find $\int_{y^2}^4 \cos(x^2)\, dx$. However, if we change the order of integration to have (Figure 15.8)

$$\int_0^2 \int_{y^2}^4 \cos(x^2)\, dx\, dy = \int_0^4 \int_0^{\sqrt{x}} y\cos(x^2)\, dy\, dx$$
$$= \int_0^4 \frac{y^2}{2}\cos(x^2)\Big|_0^{\sqrt{x}} dx$$
$$= \int_0^4 \frac{x}{2}\cos(x^2)\, dx$$
$$= \frac{1}{4}\int_0^{16} \cos u\, du = \frac{1}{4}\sin 16.$$

□